Carnap’s Philosophy of Mathematics

Benjamin Marschall

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Abstract

For several decades, Carnap’s philosophy of mathematics used to be either dismissed or ignored. It was perceived as a form of linguistic conventionalism and thus taken to rely on the bankrupt notion of truth by convention. However, recent scholarship has revealed a more subtle picture. It has been forcefully argued that Carnap is not a linguistic conventionalist in any straightforward sense, and that supposedly decisive objections against his position target a straw man. This raises two questions. First, how exactly should we characterise Carnap’s actual philosophy of mathematics? Secondly, is his position an attractive alternative to established views? I will tackle these issues by looking at Carnap’s response to the incompleteness theorems. Drawing on arguments put forward by Gödel and Beth, I show that some crucial aspects of Carnap’s positive account have remained underdeveloped. Suggestions on what a full evaluation of Carnap’s position requires are made.

1 Introduction

Unlike Frege, Hilbert, or Gödel, Carnap does not usually play a central role in textbooks or introductory courses on the philosophy of mathematics. Not because he didn’t have anything to say about this topic: in his 1934 book *The Logical Syntax of Language*, Carnap puts forward an approach to the philosophy of mathematics that is both unique and interesting.[1] But its reception has been troubled. For several decades, Carnap was widely read as a linguistic conventionalist about mathematical truth, and thus taken to fall prey to the anti-conventionalist arguments of Quine and others. Recent scholarship has challenged the conventional wisdom by showing that there is no easy refutation of Carnap’s position. But the relevant literature is not always easily accessible and tends to be read by historians of analytic philosophy only. Misconceptions about Carnap’s project therefore remain widespread.

[1] Not everyone agrees: von Neumann called Carnap’s position ”naive and feeble” (von Neumann 2005: 203).

In this paper I will first clarify the descriptive question of what Carnap's position actually is.[2] It will be helpful to proceed by asking whether the analytic/synthetic distinction and the Principle of Tolerance commit him to linguistic conventionalism. We can then move on to the evaluative question of whether Carnap's account is stable and attractive. I will discuss the two most sophisticated objections that have been put forward against it, both of which are based on the incompleteness theorems: Gödel's consistency argument and Beth's argument from non-standard models. Whether these are ultimately successful cannot be conclusively settled here. I will argue, however, that friends of Carnap's position cannot just ignore the arguments as being based on misreadings. Suggestions for future research are made.

2 Carnap and Mathematics

2.1 Analyticity and Tolerance

It would be convenient to characterise Carnap's philosophy of mathematics by relating it to more established views in the field, such as Platonism, formalism, or nominalism. But this is difficult, since the mature Carnap's approach to philosophy differs quite drastically from that of typical proponents of such views. We can start by noticing that Carnap is happy to *talk* like a Platonist. His well-known "Empiricism, Semantics, and Ontology" argues that empiricists can use languages that refer to abstract objects. Carnap thus makes Platonist-sounding claims such as

- Numbers exist (Carnap 1956a: 208).
- Numerals refer to numbers (Carnap 1956a: 216).
- Numbers are mind-independent entities.[3]

The acceptance of abstract objects is usually thought to face various philosophical challenges. Here is a representative collection:

- **Ontology:** Are abstract objects compatible with a naturalistic worldview? Are they indispensable?

[2] I will not cover Carnap's early philosophy of mathematics. See Awodey and Carus 2001 and Schiemer 2013.

[3] Technically he only says this about propositions, but the consideration put forward applies to all abstract objects (Carnap 1956a: 210f).

- **Philosophy of language:** How is reference to abstract objects possible? Causal theories of reference don't work; so what's the alternative?
- **Epistemology:** How can we know about abstract objects? Causal theories of knowledge don't work; so what's the alternative?

Carnap, however, doesn't really address these questions directly, at least not in a way that resembles mainstream discussions. To see why not, we need to introduce two core components of his position: the notion of analyticity and the Principle of Tolerance.

In order to understand Carnap's conception of analyticity, it is crucial to appreciate what he means by *languages*. In most contexts he refers to what he would later call *linguistic frameworks*: formalised languages with explicitly stated rules of use for the expressions provided. In such a language, some sentences are derivable from the rules themselves, just as logical truths can be derived in a calculus without any additional premises. These sentences are classified as *analytic*, whereas sentences that are independent of the rules are *synthetic*.[4] Empirical claims about the observable world are the paradigmatic examples of synthetic statements (Carnap 1937: 41). But analytic sentences are meaningful too, despite being inferentially isolated from statements about observations (Carnap 1937: 318f). And, importantly, Carnap sets up the languages he constructs in such a way that mathematics comes out as analytic. Mathematical questions are thus answered through "logical analysis based on the rules for the new expressions" (Carnap 1956a: 209).

One might think, however, that the analyticity of mathematics merely relocates the philosophical worries about abstract entities. For if the rules of a language allow us to infer mathematical claims, don't these rules themselves need to be *justified*? Don't we, for instance, have to base their acceptance on some prior "ontological insight" (Carnap 1956a: 214) about the nature of numbers? At this point the Principle of Tolerance – the key innovation of *The Logical Syntax of Language* – comes in:

> *In logic there are no morals.* Everyone is at liberty to build up his own logic, i.e. his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments. (Carnap 1937: 52)[5]

4 A sentence whose negation is analytic is *contradictory*.
5 Carnap construes logic broadly to also include mathematics.

According to Carnap, the widespread presupposition that the acceptance of logical and mathematical languages requires a philosophical justification is a mistake. Once clear rules for the use of the expressions of such a language have been laid down, we can freely decide whether to adopt it for some purpose or not. Philosophical arguments are out of place. The Principle of Tolerance thus allows us to avoid "pseudo-problems and wearisome controversies" in favour of a "boundless ocean of unlimited possibilities" (Carnap 1937: xv).[6]

Taken together, the notion of analyticity and the Principle of Tolerance aim to dissolve the philosophical obstacles associated with mathematics. Construed as analytic, mathematical statements follow from linguistic rules, and the adoption of such rules is a matter of pragmatic decision. Carnap thus takes himself to have shown that using a mathematical language "does not imply embracing a Platonic ontology but is perfectly compatible with empiricism and strictly scientific thinking" (Carnap 1956a: 206).

2.2 Is this Conventionalism?

Carnap was and is frequently read as a conventionalist about mathematical truth (Putnam 1979, Potter 2000: chapter 11, Warren 2020: chapter 13). And this is not surprising. Linguistic conventionalism says, roughly, that mathematical truth is determined by linguistic conventions. For Carnap, mathematical statements are analytic, which means that they follow from the rules of a certain constructed language. Furthermore, adopting such rules does not require any prior justification, but is a purely conventional choice. Taken together, these two ideas seem to amount to the conventionalist thesis.

However, linguistic conventionalism faces some influential objections. Do these undermine Carnap's position? Not in any straightforward way.[7] Consider, first, the Master Argument against the very idea of truth by convention popularised by Boghossian (Boghossian 2017). It targets a form of conventionalism that is committed to the following claim:

> Whereas conventions always determine which proposition a sentence *expresses*, in the case of logic and mathematics they also *make it the case* that the proposition expressed is true.

Boghossian objects that this thesis is absurd, because it would have been true

6 How – if at all – does Carnap justify the Principle of Tolerance *itself*? On this difficult question see Putnam 1983, Ricketts 1994, and Richardson 2007.

7 See Warren 2020 for a more general critique of common objections to conventionalism.

that *snow is either white or it isn't* even if there had been no conventions at all (Boghossian 2017: 583).

This may be a compelling argument against *some* versions of conventionalism. But is it any good against Carnap? No. Scholars have forcefully argued that Carnap's formulations should not be read in a metaphysically loaded way, since he "rejects any thick notion of truth-in-virtue-of" (Ricketts 2007: 211, see also Ebbs 2017: 25). Carnap does not endorse any claims that involve metaphysical notions like *truthmaking*, and so Boghossian's critique leaves him unaffected.

In his "Truth by Convention", Quine has given another highly influential argument against conventionalism (Quine 1949). The key charge is one of circularity. Since there are infinitely many logical truths, conventions would somehow need to settle the truth values of infinitely many sentences. But, on the other hand, the conventions need to be statable in a finite (or at least recursive) way. Infinitely many stipulated truths thus only arise if some of the conventions are general in nature, such as

> Let every instance of the following schema be true: $\ulcorner \phi \to \phi \urcorner$.

But in order to show that, say, 'q $\to$ q' is such an instance, logical rules are needed: universal instantiation and modus ponens. The relevant conventions thus *presuppose* principles of logic that they do not account for. Conventionalism, says Quine, is viciously circular.

This may be a powerful argument against *some* versions of conventionalism. But is it any good against Carnap? No. As scholars have stressed, it is implausible to read Carnap as attempting to give a *non-circular* explanation of logical (and mathematical) truth in the sense Quine has in mind (Goldfarb and Ricketts 1992: 71, Goldfarb 1995: 330). Carnap is open and upfront about the fact that his definitions of 'analytic' for the mathematical object languages can only be given in a metalanguage that is itself mathematical – and, in addition, stronger than the object language (Carnap 1937: 100, 129). Far from being unaware of Quine's point, Carnap must therefore not have aimed for non-circularity in the first place.[8]

We can conclude that there is no quick and easy way to refute Carnap's position. But the strategy of maintaining that Carnap never wanted to make claims about truthmaking, or give non-circular explanations, looks suspicious. If every objection would be dismissed in this manner, then Carnap's philosophy of math-

[8] Did Quine *intend* to target Carnap with his critique? See Ebbs 2011 and Morris 2018 for cases against this.

ematics seems to lack any positive content.[9] This is indeed an important concern. If there were *no* connection between the conventionalist idea and Carnap's position at all, it becomes difficult to understand how the analyticity of mathematics is able to dissolve philosophical concerns about abstract objects. But it is hard to make progress at such a high level of abstraction. I will therefore proceed by looking at two specific objections that cannot be dismissed without further ado: Gödel's consistency argument and Beth's argument from non-standard models.

3 Gödel's Consistency Argument

3.1 Incompleteness and Consistency

In 1953 Gödel was asked to contribute to the Schilpp volume on Carnap. Over the coming years he produced a number of drafts with the title "Is Mathematics Syntax of Language?". In 1959 Gödel decided not to have any of them published, so Carnap himself never read the critique. I will focus on the most widely discussed argument found in the drafts: the argument from consistency.[10]

Gödel's second incompleteness theorem shows that no consistent and sufficiently strong mathematical theory proves its own consistency. The consistency of Peano arithmetic (PA), for instance, cannot be demonstrated in PA itself, or in any weaker theory. Stronger mathematics is needed, such as set theory. Gödel thought that this result affects Carnap's position, which he took to be that "[m]athematical intuition, for all scientifically relevant purposes, [...] can be replaced by conventions about the use of symbols and their application" (Gödel 1995: 356).

In order to concisely state Gödel's argument, it will be helpful to rely on the notion of a *conservative extension*:

> Theory T^* is a conservative extension of a theory T iff
>
> (i) every theorem of T is a also theorem of T^* and
>
> (ii) every theorem of T^* that is expressed in the language of T is also a theorem of T.

Suppose that we currently accept a (consistent) base theory in which we can make claims about the empirical world, such as "this table is black", but which

[9] Putnam seems worried about this when writing that "the Carnap who holds no doctrines but only asks for 'clarification' [...] is just not the Carnap I knew and loved" (Putnam 1994: 281).

[10] See Lavers 2019 for other considerations Gödel puts forward.

does not yet contain mathematics. Carnap thinks that, if we want to, we could extend this theory by adding linguistic conventions for the use of mathematical symbols. Gödel challenges this assumption in the following way:

(1) If some theory is adopted as a convention, it must be known that it is a conservative extension of the base theory.

(2) A theory that extends a consistent base theory is conservative only if the former theory is consistent.

(3) So: A mathematical theory can be adopted as a convention only if it is known that this theory is consistent.

(4) The second incompleteness theorem shows that, for any sufficiently strong mathematical theory, we need even stronger mathematics to prove that theory's consistency.

(5) We therefore need to rely on mathematical intuition in order to know that the theory we want to adopt as a convention is consistent.

The step from (4) to (5) is contentious. What does Gödel mean by mathematical intuition? And how exactly is it connected to the need for stronger mathematics? But discussing this part of the argument would lead us too far afield, since those who defend Carnap have focused their attention on premise (1).[11] Let us therefore consider why Gödel accepted it.

The motivation behind (1) is a natural one. We assumed that, in the base theory, we can talk about empirical objects. If we extend it by a non-conservative theory, then new empirical claims become derivable. In particular, if we extend it by an inconsistent theory, *every* sentence is derivable, including "there are seven billion red tables in Iceland". This is a false empirical claim. And, so Gödel argues, this shows that the addition of the mathematical theory cannot be classified as conventional. Conventionality requires that nothing can *refute* the added theory. Otherwise we are dealing with a *hypothesis* rather than a convention (Gödel 1995: 339). It is therefore crucial to insist on the conservativeness of mathematics, which in turn requires a consistency proof.

3.2 Facts and Conventions

The canonical reply to Gödel's argument has been given by Goldfarb and Ricketts. According to them, Gödel's objection misses its target, since it presupposes

[11] For more on Gödelian intuition see Potter 2001 and Wrigley forthcoming.

a "language-transcendent notion of empirical fact" that Carnap rejects (Goldfarb and Ricketts 1992: 65). But it is hardly transparent what this means (Eklund 2012). So let us go through their defence using the previous example.

We supposed that the sentence

(T) There are seven billion red tables in Iceland

is expressible in, but not entailed by, the empirical base theory. Adding inconsistent mathematics then makes every sentence derivable, including (T). Gödel took this to show that violating the consistency requirement leads to false empirical claims.

Goldfarb and Ricketts reject the latter conclusion. They stress that, while the *sentence* (T) indeed becomes derivable in the inconsistent theory, we should *not* take it to make any empirical claim at all. This is because Carnap accepts a holistic definition of empirical content: the content of a sentence is the set of synthetic sentences that follow from it (Carnap 1937: 175). In an inconsistent theory, however, *every* sentence is analytic, since they can all be derived from the (inconsistent) rules. Therefore no sentence has any empirical content at all. This point can also be put in terms of the distinction between sentences and the propositions they express. Whereas (T), considered as part of the base theory, expresses an empirical proposition, every sentence of an inconsistent theory – and thus also (T) – expresses the same trivial proposition.

One can question whether this move is really a *defence* of Carnap, however, rather than the illustration of an even more serious problem. Potter has argued for the latter conclusion, by pointing out the following consequences of Carnap's position:

(1) Given Carnap's holistic conception of content, only consistent languages can make any claims about the empirical world.

(2) Our language includes mathematics.

(3) We have no a priori proof of the consistency of our language.

(4) We therefore have at most inductive empirical evidence that our language is consistent.

(4) Consequently, we have at most inductive empirical evidence that the sentences of our language make any claims about the empirical world.

Potter describes this conclusion as ”plainly absurd” and ”as close to a straightforward contradiction as one is likely to encounter in philosophy” (Potter 2000: 277). This verdict may be a bit strong. But Potter is surely right that treating the question of whether our language is about an objective empirical world at all as itself being an empirical question has the air of paradox.

3.3 Frameworks and Language

Potter's improved version of Gödel's consistency argument has not been left unchallenged. Its force crucially depends on the following unstated assumption:

> What Carnap says about formal languages in *Logical Syntax* also holds for natural languages such as English.

After all, the conclusion Potter draws is only striking – and potentially absurd – if it holds for ”our language”, i.e. some natural language. But, as I already stressed, Carnap is primarily interested in formalised frameworks and doesn't tend to make claims about natural language. In their ”How Carnap Could Have Replied to Gödel”, Awodey and Carus rely on precisely this fact to respond to Potter:

> Though [Carnap's] conception of the scientific, theoretical language was holistic [...], he specifically saw the practical, intuitive part of language as distinct from it and serving a different purpose. (Awodey and Carus 2004: 213)

According to Awodey and Carus, the apparently absurd conclusion Potter draws is avoided because ”the purely descriptive sentences [of ordinary language] remain unaffected by an inconsistency in the theoretical language [...] since the interpretation of [ordinary language] is given by practical agreement in its use” (Awodey and Carus 2004: 214n19). Here they follow Carnap's own remark that the language we use to talk about observable things has a ”complete interpretation” because it ”is used by a certain language community as a means of communication, and [...] all sentences of [it] are understood by all members of the group in the same sense” (Carnap 1956b: 40).

”Our language” should thus not be equated with a Carnapian framework. In addition, the holistic conception of content that makes inconsistent languages unable to express empirical claims only pertains to the latter. Contra Potter, we therefore cannot draw any conclusions about natural language from Gödel's argument.

What are we to make of this reply? It is hard to be fully satisfied, since it gives rise to the following question: How exactly does our use of ordinary language manage to give it an empirical interpretation?[12] In his mature work, Carnap usually takes the observational part of language for granted and focuses his attention on the development of frameworks for the theoretical part (Carnap 1956b). Without a more detailed account of how ordinary language makes contact with the world, however, a crucial element of Awodey and Carus's defence of Carnap remains an unexplained black box.[13]

Neither Gödel's original consistency argument nor Potter's improved version has delivered a straightforward refutation of Carnap's position. But they show that, even though Carnap himself is primarily interested in the development of formalised framework, the role of ordinary language is nevertheless crucial to assess the viability of his position. A similar lesson will emerge from the following discussion of Beth.

4 Beth's Argument from Non-Standard Models

4.1 Incompleteness and Analyticity

A paper by Beth that was actually published in the Schilpp volume contains a second argument based on the incompleteness theorems. It will be useful to state the first theorem in more detail:

No theory T has all of the following properties:

(1) T is consistent
(2) T is recursively formalised[14]
(3) T is strong enough to do basic arithmetic
(4) T is complete, i.e. for any T-sentence ϕ, either T proves ϕ or T proves $\neg\phi$

In order to understand Beth's argument, we then need to take a closer look at the definitions of 'analytic' Carnap gives in *Logical Syntax*. His aim is to classify every true mathematical sentence as analytic and every false one as contradictory.

[12] There also seems to be a tension between the current strategy and Carnap's insistence, in a reply to Strawson, that there is no in-principle difference between natural and formal languages (Carnap 1963: 938).

[13] It is an interesting question whether Carnap could help himself to the account of scientific objectivity Quine started to develop in the 1950s (Quine 1957, Quine 1960).

[14] This means that T's axioms can be recursively enumerated and that its inference rules are decidable (Raatikainen 2020: section 1.1).

In other words, the definition of 'analytic' is supposed to be *complete*, leaving the status of no purely mathematical sentence undetermined. But how, in light of the incompleteness result just stated, is this even possible?

The task would be hopeless if one defined analyticity in terms of what is derivable from a recursively formalised theory. But Gödel's theorems don't apply to *non-recursive* theories, and Carnap makes use of this fact. For Language I of *Logical Syntax*, he relies on the infinitary ω-rule (Carnap 1937: 38, 173):

ω-Rule

$$\frac{\phi(0), \phi(1), \phi(2), ...}{\forall x \phi(x)}$$

For Language II, Carnap's definition is a version of a Tarskian truth-theory, even though it is still presented as a syntactic definition (Carnap 1942: 247, Coffa 1987, Koellner 2009). More specifically, Carnap characterises analyticity for Language II by relying on a domain of quantification containing the accented expressions 0, $0'$, $0''$, ... (Carnap 1937: 106). These are assumed to be isomorphic to the natural numbers, and so analyticity is equated with truth in the standard model of arithmetic ($\mathbb{N}$).[15] In modern terminology Carnap's definition thus amounts to the following:

ϕ is analytic in Language II **iff** $\mathbb{N} \vDash \phi$.

Beth takes issue with Carnap's reliance on an infinite stock of accented expressions. After rightly stressing that these expressions need to correspond to the standard numerals, Beth points out – also rightly – that enumerating some instances does not suffice to pin down the intended interpretation. Beth illustrates this by introducing the fictitious logician Carnap*, who thinks that there are *more* accented expressions than standard numerals. This character therefore equates analyticity in Language II with truth in a *non-standard* model of arithmetic.

Beth clearly thought that the possibility of someone like Carnap* is a problem for Carnap's position. But his paper is rather elusive about what exactly this problem is supposed to be. In the secondary literature, two readings of Beth can be distinguished. One puts the emphasis on the apparent *model-theoretic* character of the argument, while the other stresses the connection to considerations about *rule-following*. We will discuss these in turn.

[15] Carnap arrived at this semantic definition of analyticity after Gödel had criticised an earlier, more straightforwardly syntactical, proposal (Gödel 2003: 342-359, Goldfarb 2003, Flocke 2019).

4.2 The Skolemite Reading

After introducing Carnap*, Beth writes that "the above considerations [...] are only variants of the Löwenheim-Skolem paradox" (Beth 1963: 478). Ricketts takes this allusion very seriously. One interesting consequence of the Löwenheim-Skolem theorem is that Zermelo-Fraenkel set theory (ZFC) can be interpreted in a *countable* model. Skolem thought that there was something paradoxical about this, since ZFC proves that there is an *un*countable set.[16] According to Ricketts, Beth's point is that the same observation applies to the language in which *Logical Syntax* is written itself – call this the *syntax language*. While Carnap intends the arithmetical part of the syntax language to be interpreted in the (countable) standard model of arithmetic, another reader – like Carnap* – might interpret it as being about a (possibly uncountable) non-standard model.

If this is Beth's argument, however, then Carnap has little reason to be concerned. For what is so problematic about the availability of non-standard interpretations in itself? One might take Skolem's observation to imply that ZFC does not "really" prove that there are uncountable sets. But, from Carnap's perspective, this philosophical gloss on the model-theoretic result can be rejected. He would say the following:

> [...] the distinction between countable and uncountable sets is an internal distinction, a distinction within our mathematical theory. [...] It makes no sense to wonder whether sets really are countable in some absolute sense, apart from the sense we, following Cantor, give this term in the context of our mathematics. (Tymoczko 1989: 290).

After all, the theory ZFC is in effect a Carnapian framework, and hence the question of whether there are uncountable sets, when asked relative to this framework, receives an affirmative answer. The same move can then be made for the language of *Logical Syntax*. If Beth's intention was to show that this syntax language is not "really" about syntax, then we have no compelling case.

However, there is reason to think that the model-theoretic reconstruction does not capture the argument Beth actually had in mind. Ricketts writes that the divergence between Carnap and Carnap* "need not be manifest in their use of the sentences of the informal syntax language" (Ricketts 2004: 194). But in Beth's own example they *do* disagree in this way. In particular, Carnap* challenges one of Carnap's results from *Logical Syntax*: that the consistency sentence of Language II is analytic but not provable (Carnap 1937: 133, Beth 1963: 478,

16 See Bays 2014 for the technical and philosophical background.

480f). This only makes sense if Carnap*'s non-standard model is such that it makes this consistency sentence false, whereas the standard model makes it true. Hence there is disagreement about at least one sentence of the syntax language.

Beth himself introduced the analogy to Skolem's paradox, but it does not actually fit the way his reasoning proceeds. For the Löwenheim-Skolem theorems only establish the existence of *elementarily equivalent* non-standard models, which make the very same sentences true as the standard model. That a non-standard model of the kind Carnap* relies on exists, on the other hand, follows from the incompleteness theorems (Beth 1963: 477). Unlike Ricketts' Skolemite reading, the rule-following interpretation of Beth make essential use of this fact.

4.3 The Rule-Following Reading

Beth writes that the case of Carnap* demonstrates a "limitation regarding the Principle of Tolerance" (Beth 1963: 479). On the rule-following reading, the relevant limitation concerns our ability to actually *use* Carnap's Languages I and II. The problem, so the basic idea, is that using them would require the ability to follow infinitary rules, which we human beings do not have.

At this point it is important to appreciate the *voluntarist* character of Carnap's metaphilosophy (Jeffrey 1994, Richardson 2007). What, one may ask, does Carnap want to achieve with the formal languages he develops in *Logical Syntax*? They are meant as *proposals* for adoption, to be used by, for instance, scientists in order to formulate physical theories.[17] In accordance with the Principle of Tolerance, it makes no sense to criticise Languages I and II as being incorrect. One can, however, argue that these languages are not *useful* for their intended purpose. One especially severe instance of such a critique would be to maintain that the rules of these languages are *impossible* to follow in practice, even if one wanted to. On the current reading of Beth, he makes precisely this claim about Carnap's definitions of 'analytic' for mathematics.

Friedman has previously argued that Carnap's reliance on infinitary rules is problematic (Friedman 1999a, Friedman 1999b). Ebbs has previously argued that Beth's argument concerns the adoption of Carnapian languages for prac-

[17] For Carnap, the task of philosophy is thus not the *analysis* of already given concepts, but rather their *explication*: the development of new and better concepts to replace old and imprecise ones (Carnap 1950: chapter 1). Interest in this approach has recently been revived by the *conceptual engineering* movement (Plunkett and Sundell 2013, Burgess and Plunkett 2013a, Burgess and Plunkett 2013b, Cappelen 2018, Burgess et al. 2020). Flocke 2020 and Kraut 2020 argue that Carnap's voluntarism leads to a form of *non-cognitivism* about theoretical philosophy.

tical use (Ebbs 1997: sections 60-61). The connection between these two ideas has recently been drawn by Marschall (Marschall 2021). It is easiest to appreciate the potential problem for Language I, in which Carnap characterises the analytic sentences by means of the infinitary ω-rule. He writes that ”there is nothing to prevent the practical application of such a rule” (Carnap 1937: 173). But the prevailing consensus in the philosophy of mathematics denies this, and takes only recursive inference rules to be followable (McGee 1991, Field 1994, Raatikainen 2005, Button and Walsh 2018: chapter 7). If this is correct, then no one can use actually use the rules of Language I – which arguably means that, qua proposal, Language I is a failure.

The situation may seem better for Language II and its semantic definition of analyticity. But this is illusory. As we have seen, Carnap assumes that Language II is interpreted in the standard model of arithmetic. He furthermore holds that ”one of the semantic rules specifies the universe of discourse” (Frost-Arnold 2013: 77, Carnap 1939: 32). But without further clarification it is not clear what this even means. In what senses and by what means did Carnap specify the universe of discourse of Language II to only contain standard numerals, and not also the non-standard elements of Carnap*'s interpretation? It is tempting to think of domain specification as an ostensive act by means of which objects to quantify over are singled out. But this cannot be the right model for abstract objects like numerals or numbers. We thus lack a clear conception of what the adoption of Language II amounts to.

Nothing Carnap says in his own reply to Beth can ameliorate the concerns about infinitary rule-following (Carnap 1963: section 18). Two different kinds of responses are possible. One is going on the offensive and denying that there is any problem about adopting the infinitary rules Carnap uses. For the special case of the ω-rule one can draw on Warren's recent work which challenges the conventional wisdom about the rule's alleged unusability (Warren 2020, Warren forthcoming). The second kind of response denies the underlying assumption that we need to follow the rules for analyticity in any substantial sense. This is suggested by Ricketts at one point (Ricketts 2003: 262) and will appeal to those who stress the quietist elements in Carnap's philosophy. One open question is whether, given this route, one can make sense of the importance Carnap clearly assigns to the notion of analyticity. We are thus led back to the question about the relationship to conventionalism from the end of section 2.2: What role do linguistic rules really play in avoiding the philosophical problems raised by mathematical objects?

Once again, we see that looking at the formal systems Language I and II in itself is not enough to judge the success of Carnap's project. Much depends on whether and how one can use these systems in practice, which in turn depends on the abilities of human beings. Questions about the relationship between ordinary language and formal systems – such as what it means to adopt certain rules – thus need to be put on the agenda.

5 Conclusion

What, then, is Carnap's philosophy of mathematics? The discussion of Gödel and Beth demonstrates that an account of *language use* is required to fully answer this question. It is essential to understand how the non-mathematical part of language gets an empirical interpretation, and to see what it means to adopt linguistic rules in practice. Unfortunately this is a topic Carnap himself said relatively little about. Important work on this issue has been published in recent years (Stein 1992, Ricketts 2003, Carus 2007, Carus 2017), but its implications for Carnap's philosophy of mathematics have not been widely explored so far. I suggest that this is the most fruitful path forward for those interested in Carnap's approach.

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