Conventionalism in Logic and Mathematics

Benjamin Marschall

bm515@cam.ac.uk

Lecture 4: Wittgenstein and Radical Conventionalism

Overview

Plan of the four lectures:

1. Conventionalism: What, why, and how?
2. Quine against Truth by Convention
3. Gödel on Convention and Consistency
4. **Wittgenstein and Radical Conventionalism**

Structure

A Look Back at Quine

- In the lecture on Quine we saw that a conventionalist needs to find some way to explain how the truth and falsity of infinitely many sentences could be settled by conventions.
- The general strategy is to start with some finite set of axioms and rules, such that the rules allow the *derivation* of infinitely many sentences.
- Quine's worry is that there is no good way to state these initial rules, but for this lecture let's consider this problem to be solved and set it aside.

A Look Back at Quine

The general picture is thus as follows:

$$\text{Initial conventions} \xrightarrow[\text{determine truth-values of}]{} \text{Infinitely many sentences}$$

- Here's something that's arbitrary and a matter of choice on this picture: which initial conventions to adopt.
- But here's something that doesn't seem to be a matter of convention: what follows once the initial conventions have been adopted.

An Example

- The conventionalist will say that whether we adopt classical or intuitionist logic as a matter of convention.
- But consider:
  (1) Classical logic entails the law of excluded middle.
  (2) Intuitionist logic does not entail the law of excluded middle.
- (1) and (2) seem to be objective facts *about* conventions, and their truth doesn't seem to be a matter of convention itself.
- Is this a problem?

Another Look at Gödel

Some body of unconditional mathematical truth must be acknowledged, because even if mathematics is to be interpreted to be a hypothetico-deductive system, still the proposition which states that the axioms imply the theorems must be unconditionally true. [...]

Another Look at Gödel

Some body of unconditional mathematical truth must be acknowledged, because even if mathematics is to be interpreted to be a hypothetico-deductive system, still the proposition which states that the axioms imply the theorems must be unconditionally true. [...] For while the definitions of 5, 7, 12 and the rules of computation for $+$ and $=$ [...] seemingly can be interpreted as conventions, the statement that $5+7=12$ follows from these conventions evidently expresses an objective (combinatorial) fact (Gödel 1995: 200).

Another Look at Gödel

- Here's a way to construe this as an objection to conventionalism.
- The conventionalist wants to explain mathematics in terms of conventions. Part of the motivation is that more realist views – such as mathematical Platonism – seem metaphysically dubious.
- But the conventionalist needs to accept certain facts about syntax, namely facts about what certain conventions entail or don't entail.
- But if there's a problem with objective facts about mathematics, facts about entailment are no better off.
- The alleged metaphysical advantages of conventionalism are thus illusory.

Syntax and Arithmetic

- Indeed, this point relates to Gödel's consistency argument.
- Consider the consistency sentence $Con_{PA} =_{def} \neg\exists x Pr_{PA}(x, \ulcorner \bot \urcorner)$
- The Pr_{PA} expresses a claim about syntax in the language of arithmetic.
- $Pr_{PA}(a, b)$ is true iff a encodes a proof of the formula encoded by b.
- That we can express claims about syntax in the language of arithmetic supports the earlier point that making sense of syntax is not any easier than making sense of mathematics.

Syntax and Arithmetic

- If Gödel is right, conventionalists must become more modest.
- They cannot uphold the initial promise of making all metaphysical complications concerning mathematics go away.
- At least a basic part of mathematics – namely arithmetic, which can encode syntax – needs to be explained in a non-conventionalist way.
- Is there any way around this conclusion? – Maybe.

Structure

Wittgenstein and Dummett

Michael Dummett

Wittgenstein according Dummett

In his "Wittgenstein's Philosophy of Mathematics", Dummett makes the point that the standard form of conventionalism doesn't really solve any problems:

> *This account is entirely superficial and throws away all the advantages of conventionalism, since it leaves unexplained the status of the assertion that certain conventions have certain consequences.*

The alternative(?):

> *Wittgenstein goes in for a* **full-blooded conventionalism***; for him the logical necessity of any statement is always the direct expression of a linguistic convention. (Dummett 1959: 328f)*

Radical Conventionalism

Let's try to understand this full-blooded (or *radical*) conventionalism by looking at what proofs achieve on this conception.

> *We naturally think that, face to face with a proof, we have no alternative but to accept the proof if we are to remain faithful to the understanding we already had of the expressions contained in it.*

Radical Conventionalism

Let's try to understand this full-blooded (or *radical*) conventionalism by looking at what proofs achieve on this conception.

> *We naturally think that, face to face with a proof, we have no alternative but to accept the proof if we are to remain faithful to the understanding we already had of the expressions contained in it. For Wittgenstein, accepting the theorem is adopting a new rule of language, and hence our concepts cannot remain unchanged at the end of the proof. (Dummett 1959: 332f)*

Radical Conventionalism

But we could have rejected the proof without doing any more violence to our concepts than is done by accepting it; in rejecting it we could have remained equally faithful to the concepts with which we started out.

Radical Conventionalism

But we could have rejected the proof without doing any more violence to our concepts than is done by accepting it; in rejecting it we could have remained equally faithful to the concepts with which we started out. *It seems extraordinarily difficult to take this idea seriously when we think of some particular actual proof. It may of course be said that this is because we have already accepted the proof and thereby subjected our concepts to the modification which acceptance of the proof involved; but the difficulty of believing Wittgenstein's account of the matter while reading the proof of some theorem with which one was not previously familiar is just as great. (Dummett 1959: 333)*

Radical Conventionalism

Dummett's Interpretation

[...] at each step we are free to choose to accept or reject the proof; there is nothing in our formulation of the axioms and of the rules of inference, and nothing in our minds when we accepted these before the proof was given, which of itself shows whether we shall accept the proof or not; and hence there is nothing which *forces* us to accept the proof. (Dummett 1959: 330)

- This form of conventionalism certainly doesn't have the problem that there are objective facts about entailment that are unaccounted for.
- But, on the other hand, it seems to be quite an outlandish view with little independent motivation.

Stroud versus Dummett

Barry Stroud has challenged Dummett's interpretation. Putnam provides a useful summary:

> *In response, Barry Stroud pointed out that the position Dummett calls, "radical conventionalism" cannot possibly be Wittgenstein's. A convention, in the literal sense, is something we can legislate either way. Wittgenstein does not anywhere say or suggest that the mathematician proving a theorem is legislating that it shall be a theorem (and the mathematician would get into a lot of trouble, to put it mildly, if he tried to "legislate" it the opposite way). (Putnam 1979: 424f)*

Stroud versus Dummett

Basing himself on a good deal of textual evidence, Stroud suggested that Wittgenstein's position was that it is not convention or legislation but our **forms of life** *(i.e.,* **our human nature as determined by our biology-plus-cultural-history***) that cause us to accept certain proofs as proofs. (Putnam 1979: 424f)*

Stroud versus Dummett

Stroud sees his proposal as an alternative to both mathematical Platonism and Dummett's radical conventionalism:

> *It is [...] a contingent fact that calculating, inferring, and so forth, are carried out in the ways that they are-just as it is a contingent fact that there is such a thing as calculating or inferring at all.*

Stroud versus Dummett

Stroud sees his proposal as an alternative to both mathematical Platonism and Dummett's radical conventionalism:

> *It is [...] a contingent fact that calculating, inferring, and so forth, are carried out in the ways that they are-just as it is a contingent fact that there is such a thing as calculating or inferring at all. But we can understand and acknowledge the contingency of this fact, and hence the possibility of different ways of calculating, and so forth, without understanding what those different ways might have been. If so, then it does not follow that those rules by which calculating, and so forth, might have been carried out constitute a set of genuine alternatives open to us among which we could choose, or even among which we could have chosen. (Stroud 1965: 513)*

Back to Entailment

$$\text{Initial conventions} \xrightarrow[\text{determine truth-values of}]{} \text{Infinitely many sentences}$$

- Gödel's point: facts about entailment are objective, so there is no way around Platonistic facts.
- Dummett's Wittgenstein: there are no facts about entailment.
- Stroud's Wittgenstein: there are facts about entailment, but they are not primitive and eternal, but rather explained by our contingent forms of life.

Structure

Putnam versus Stroud

- Does Stroud's Wittgenstein do the trick?
- Putnam argues: *no*, facts about consistency still need to be construed as *objective mathematical facts*.
- We end the lecture by looking at this argument.

Unpacking Forms of Life

- If Stroud is right, we must make sense of the claim that our forms of life make it the case that – for instance – Peano arithmetic is consistent.
- What are forms of life? Following Putnam, let's say they roughly speaking are *dispositions to use mathematical expressions* in certain ways.
- Underlying idea: our mathematical *practice* determines the interpretation of mathematical language.

Unpacking Forms of Life

[...] human practice, actual and potential, only extends finitely far. We cannot "go on counting forever" – even if we say we can, not really. If there are possible divergent extensions of our practice, then there are possible divergent interpretations of even the natural member sequence-our practice, our mental representations, etc., do not (in set theoretic terminology) single out a unique "standard model" of even the natural number sequence.

Unpacking Forms of Life

[...] human practice, actual and potential, only extends finitely far. We cannot "go on counting forever" – even if we say we can, not really. If there are possible divergent extensions of our practice, then there are possible divergent interpretations of even the natural member sequence-our practice, our mental representations, etc., do not (in set theoretic terminology) single out a unique "standard model" of even the natural number sequence.
[...] for the same reason, "Peano arithmetic is consistent" may have no truth value – for this statement too talks about an infinite sequence (the sequence of all theorems of Peano arithmetic), and the sequence may not really be determinate. (Putnam 1979: 426)

Varieties of Consistency

- Wittgenstein is in a position to deny that "Peano arithmetic is consistent" has a determinate truth value at all.
- This is because the truth of this claim requires that *none* of the infinitely many theorems of PA is a contradiction.
- But our practice is finite, and – according to Putnam – doesn't settle the truth value of the consistency statement.

Varieties of Consistency

- Wittgenstein is in a position to deny that "Peano arithmetic is consistent" has a determinate truth value at all.
- This is because the truth of this claim requires that *none* of the infinitely many theorems of PA is a contradiction.
- But our practice is finite, and – according to Putnam – doesn't settle the truth value of the consistency statement.

Still, assuming some number – say 10^{20} *– is small enough so that we could collectively and over time (perhaps several generations) examine all proofs with fewer than that number of symbols, the question "Is Peano arithmetic* 10^{20}*-consistent?" should have a determinate answer even on Wittgenstein's view. (Putnam 1979: 426)*

Explaining Consistency

- The crucial question now is: can one plausibly maintain that our forms of life determine whether Peano arithmetic is 10^{20}-consistent?
- Putnam thinks that there is a way to make sense of this claim. Suppose our dispositions were as follows:

Scenario A

When given a mathematical proof, we check it line by line. For each line, we make sure that it is either an axiom, or the result of applying modus ponens to two previous lines. There is one exception though: if one line reads "1=0" (or any other contradiction), we reform the rules for modus ponens so as to make this line underivable.

Explaining Consistency

Scenario A

When given a mathematical proof, we check it line by line. For each line, we make sure that it is either an axiom, or the result of applying modus ponens to two previous lines. There is one exception though: if one line reads "1=0" (or any other contradiction), we reform the rules for modus ponens so as to make this line underivable.

- If this were our forms of life, they would guarantee that Peano arithmetic is consistent.
- Inconsistency just is the provability of a contradiction, and that possibility is excluded by our way of handling proofs.

Explaining Consistency

Scenario B

When given a mathematical proof, we check it line by line. For each line, we make sure that it is either an axiom, or the result of applying modus ponens to two previous lines. If one line reads "1=0" (or any other contradiction), **we announce that Peano arithmetic is inconsistent.**

- But our actual dispositions are arguably like those in scenario B, not A.
- And in this case our dispositions don't settle the question of consistency at all.
- Whether we ever reach a contradiction or not depends on the nature of the axioms and inference rules under consideration.
- And with this we are back where we started from.

Explaining Consistency

Scenario B

When given a mathematical proof, we check it line by line. For each line, we make sure that it is either an axiom, or the result of applying modus ponens to two previous lines. If one line reads "$1=0$" (or any other contradiction), **we announce that Peano arithmetic is inconsistent.**

> *[...] consistency is an objective mathematical fact, not an empirical fact. Thus, there is at least one mathematical fact – namely the consistency of the meaning determinations themselves, whatever these be produced by – which is nor explained by our nature or "forms of life" in any intelligible sense. (Putnam 1979: 425)*

Dummett, M. (1959): 'Wittgenstein's Philosophy of Mathematics'. *Philosophical Review*, 68(3):324–348.

Gödel, K. (1995): 'Is Mathematics Syntax of Language?' In: Rodriguez-Consuegra, F. (ed), *Kurt Gödel: Unpublished Philosophical Essays*, pp. 171–222. Birkhäuser Basel.

Putnam, H. (1979): 'Analyticity and Apriority: Beyond Wittgenstein and Quine'. *Midwest Studies in Philosophy*, 4(1):423–441.

Stroud, B. (1965): 'Wittgenstein and Logical Necessity'. *Philosophical Review*, 74(October):504–518.