Conventionalism in Logic and Mathematics

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Lecture 3: Gödel on Convention and Consistency

Overview

Plan of the four lectures:

1. Conventionalism: What, why, and how?
2. Quine against Truth by Convention
3. **Gödel on Convention and Consistency**
4. Wittgenstein and Radical Conventionalism

Gödel

Kurt Gödel

Structure

Back to Mathematics

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- Last time we focused on conventionalism about *logic*.
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- Today we look at conventionalism about *mathematics*, since it poses some new problems that don't arise for logic.
- In order to appreciate these additional problems we need to look at Gödel's famous incompleteness theorems.

Completeness

- Some of you might (vaguely) remember this from the Metatheory textbook:

We say that a formal proof system is COMPLETE (relative to a given semantics) *iff* whenever Γ entails C, there is some formal proof of C whose assumptions are among Γ. To prove that TFL's proof system is complete, I need to show:

Theorem 6.1. Completeness. For any sentences Γ and C: if $\Gamma \vDash C$, then $\Gamma \vdash C$.

- We can show that TFL is complete, i.e. that every logically true sentence is derivable in the proof system we set up.
- One can also show this for FOL (this was in fact proved by Gödel).

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- One might now think that mathematics is complete, just like logic.
- But it isn't!
- The upshot of Gödel's incompleteness theorem is that we cannot find a system of axioms from which all mathematical truths are deducible.

Peano Arithmetic (PA)

(1) $\forall x(0 \neq Sx)$

(2) $\forall x \forall y(Sx = Sy \rightarrow x = y)$

(3) $\forall x(x + 0 = x)$

(4) $\forall x \forall y(x + Sy = S(x + y))$

(5) $\forall x(x \times 0 = 0)$

(6) $\forall x \forall y(x \times Sy = (x \times y) + x)$

(I) $((\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(Sx))) \rightarrow \forall x \phi(x))$

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- $PA \nvdash G$
- $PA \nvdash \neg G$
- G is a purely mathematical sentence.
- One can argue that G must be true in the intended model arithmetic, for which we write $\mathbb{N} \vDash G$.
- So: $\mathbb{N} \vDash G$ but $PA \nvdash G$ – PA is incomplete.

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- $Con_{PA} =_{def} \neg\exists x Pr_{PA}(x, \ulcorner\bot\urcorner)$
- Con_{PA} encodes the claim that no contradiction is derivable from *PA* *within* PA.
- Gödel shows that Con_{PA} is independent of PA – so it cannot prove its own consistency.

The Incompleteness Theorems

- We will not go more into the technical details here since there is a whole series on lectures on that.
- Instead we move on to Gödel's arguments against conventionalism that rely on his incompleteness results.

Structure

Gödel versus Carnap

- In the 1950s, Gödel produced a series of drafts of a paper called "Is Mathematics Syntax of Language?".
- The argumentative goal was to refute Carnap's conventionalist philosophy of mathematics as presented in his *Logical Syntax of Language*.

The Target as Described by Gödel

I. Mathematical intuition, for all scientifically relevant purposes, in particular for drawing the conclusions as to observable facts occurring in applied mathematics, can be replaced by conventions about the use of symbols and their application.

II. In contradistinction to the other sciences, which describe certain objects and facts, there do not exist any mathematical objects or facts. Mathematical propositions, because they are nothing but consequences of conventions about the use of symbols and, therefore, are compatible with all possible experiences, are void of content. (Gödel 1995: 356)

Gödel's Consistency Argument

The following definition will come in handy:

Conservative Extensions

Theory $T*$ is a conservative extension of a theory T iff

(i) every theorem of T is a also theorem of $T*$ and
(ii) every theorem of $T*$ that is expressed in the language of T is also a theorem of T.

Intuitive idea: a conservative extension doesn't let us derive anything substantially new.

First Half of Gödel's Consistency Argument

(1) If some theory is adopted as a convention, it must be known that it is a conservative extension of the base theory.

(2) A theory that extends a consistent base theory is conservative only if the former theory is consistent.

(3) So: A mathematical theory can be adopted by convention only if it is known whether this theory is consistent.

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Why should we accept (1)?

First Half of Gödel's Consistency Argument

Moreover a rule about the truth of sentences can be called syntactical only if it is clear from its formulation, or if it somehow can be known beforehand, that it does not imply the truth or falsehood of any "factual" sentence (i.e., one whose truth, owing to the semantical rules of the language, depends on extralinguistic facts). This requirement [...] follows from the concept of a convention about the use of symbols, [...]. The requirement under discussion implies that the rules of syntax must be demonstrably consistent, since from an inconsistency **every** *proposition follows, all factual propositions included. (Gödel 1995: 339)*

First Half of Gödel's Consistency Argument

- Conventions need to be arbitrary: adopting them must be a mere matter of convenience, and should not be refutable.
- From am inconsistent convention *everything* follows.
- So if we start with a theory in which we can talk about things being coloured and add an inconsistent mathematical theory, we can now derive the false claim that *grass is red*.
- This gives us a non-pragmatic reason to reject the initial convention we started with.
- Non-conservative extensions are thus no conventions but additional theoretical assumptions.

Second Half of Gödel's Argument

(4) We know from Gödel's incompleteness theorems that for any sufficiently strong mathematical theory we need even stronger mathematics to prove the theory's consistency.

(5) So we need to rely on mathematical intuition at some point in order to know that the theory we want to adopt by convention is consistent.

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Why think that a *mathematical intuition* is the only way of acquiring knowledge about consistency?

Second Half of Gödel's Argument

[...] it may be argued that, although transfinite mathematical axioms clearly must not be used, it is permissible to use empirical induction. E.g., consistency might be based on the fact that no contradiction has arisen so far. Now it is true that, if consistency is interpreted to refer to the handling of physical symbols, it is empirically verifiable like a law of nature. However, if this empirical consistency is used, mathematical axioms and sentences completely lose their "conventional" character, their "voidness of content" and their "apriority" [...] and rather become expressions of empirical facts. (Gödel 1995: 342)

Second Half of Gödel's Argument

- Why is that? The argument in Gödel's paper is not straightforward.
- I think the though is the following:
- Facts about consistency are themselves mathematical facts.
- If we use empirical induction to establish consistency, then empirical evidence must be relevant to mathematical truth.
- In that case, consistency is a substantial thesis about the way the world is after all – and so at least one mathematical fact is more than a mere convention.

Gödel's Consistency Argument

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(3) So: A mathematical theory can be adopted by convention only if it is known whether this theory is consistent.
(4) We know from Gödel's incompleteness theorems that for any sufficiently strong mathematical theory we need even stronger mathematics to prove the theory's consistency.
(5) So we need to rely on mathematical intuition at some point in order to know that the theory we want to adopt by convention is consistent.

Structure

Defensive Response I

(1) If some theory is adopted as a convention, it must be known that it is a **conservative extension** of the base theory.

- Gödel assumes that in from inconsistent theory we can derive false empirical claims, such as *grass is red*.
- But this is not what Carnap would actually say.

Defensive Response I

Empirical Content

The empirical content of a sentence S is the set of non-valid sentences which follows from S. (adapted from Carnap 1937: 175 (§49)).

- Consequence: in an inconsistent theory *no* sentence has any empirical content.
- Suppose T is a consistent empirical theory in which the sentence "grass is red" is false.
- If we extend T by adding inconsistent mathematics the sentence "grass is red" will be derivable, but it will not actually say that grass is red.

Defensive Response I

- Fair enough: we can reject the assumption that inconsistent theories make false claims about the world.
- But is this really a *defence* of Carnap?

Imagine the case of a language which has been used for many years with apparent success despite the existence within its mathematical part of an abstruse and as yet undiscovered contradiction (the Burali-Forti paradox, say). We seem to be forced by Carnap's account to say that despite appearances this language is not succeeding in saying anything about the world. (Potter 2000: 276)

Defensive Response II

(1) If some theory is adopted as a convention, **it must be known** that it is a conservative extension of the base theory.

- Why this epistemological requirement?
- Doesn't it suffice that the theory is *actually* consistent?

Defensive Response II

In short, where Gödel says "the rules of syntax must be **demonstrably** *consistent, since from an inconsistency every proposition follows" (ibid.), he should correctly say "the rules of syntax must be consistent, since from an inconsistency every proposition follows". (Awodey and Carus 2004: 208, my emphasis in bold)*

Defensive Response II

> *In short, where Gödel says "the rules of syntax must be* **demonstrably** *consistent, since from an inconsistency every proposition follows" (ibid.), he should correctly say "the rules of syntax must be consistent, since from an inconsistency every proposition follows". (Awodey and Carus 2004: 208, my emphasis in bold)*

(1*) If some theory is adopted as a convention, **it must be** a conservative extension of the base theory.

Based on (1*) we can only argue that conventions must be consistent, not that they must be known to be so.

Defensive Response II

- Ok fine. But we still need some way to know about consistency, for instance when we *apply* mathematics.

  [...] in order to have any reason for the expectation that, if these rules are applied to verified laws of nature (e.g., the primitive laws of elasticity theory), one will obtain empirically correct propositions (e.g., about the carrying power of a bridge), one evidently must know certain facts (at least with probability) concerning the rules of syntax. For to expect this for perfectly arbitrary rules about the truth or falsehood of propositions clearly would be folly. (Gödel 1995: 357)

- Furthermore: how does the conventionalist explain facts about consistency?

Offensive Response

- How could the conventionalist explain the truth of Con_{PA} even though it is not derivable from PA?
- Many options, but one idea is to strengthen PA by adding a new rule.

ω-rule

$$\frac{\phi(0),\phi(1),\phi(2),\ldots}{\forall x \phi(x)}$$

- In PA plus the ω-rule we can derive Con_{PA}.
- This seems like good news for the conventionalist, since it fits the general scheme of explaining mathematical truths in terms of inference rules.

Offensive Response

- Carnap seems to have thought that this is the way to go:

  Tarski discusses [... the ω-rule] and rightly attributes to it an "infinitist character". In his opinion: "it cannot easily be harmonized with the interpretation of the deductive method that has been accepted up to the present"; and this is so far as this rule differs fundamentalle from the [... finitary rules] which have hitherto been exclusively used. **In my opinion however, there is nothing to prevent the practical application of such a rule.** *(Carnap 1937: 173)*

Offensive Response

- Carnap's last claim is questionable.
- The ω-rule seems to violate what has been called the Cognitive Constraint: "humans cannot be attributed non-computational causal powers" (Warren and Waxman forthcoming)

  Human beings are products of nature. They are finite systems whose behavioral responses to environmental stimuli are produced by the mechanical operation of natural forces. Thus, according to Church's Thesis, human behavior ought to be simulable by a Turing machine. This will hold even for idealized humans who never make mistakes and who are allowed unlimited time, patience, and memory. (McGee 1991: 117)

- It is thus hard to see in what sense humans could be said to 'follow the ω-rule', even implicitly.

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